Physics 798C
Superconductivity
Spring 2024
Homework 5
Due March 14, 2024

1. We shall examine a **"BCS Lite" theory of superconductivity**. The point of this problem is to carry out the BCS calculation with a simple model that allows you to carry out all of the summations analytically, without ever having to do an integral on energy. Consider a model of a narrow-band superconductor in which $\hbar\omega_c$ is much larger than the width of the conduction band. You may neglect the differences in one-electron energies in the band (ε_k = constant) and treat the problem as one with N electrons in a band of zero width containing $2M$ states (the two represents spins). Hint: Choose the chemical potential to be at the energy ε_k (this simplifies the expression for the kinetic energy!)

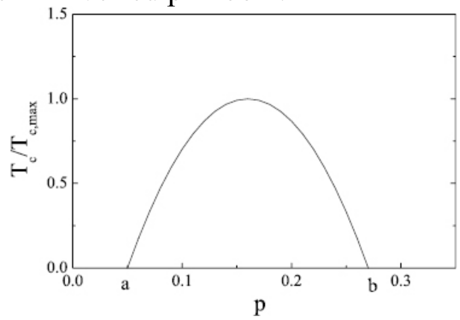
a) Start by writing down the BCS ground state wavefunction, the pairing Hamiltonian (and the expectation value of the Landau potential in terms of the u's and v's from Lecture 9), normalization constraint on the u's and v's, the average number of particles constraint ($N = 2\sum_{k_1}^{k_M} |v_k|^2$), and the self-consistent gap equation (both zero temperature $\Delta_k = -\sum_{l_1}^{l_M} V_{k,l}\, u_l\, v_l$, and finite temperature $\Delta_k = -\sum_{l_1}^{l_M} V_{k,l}\, \frac{\Delta_l}{2E_l} \tanh(\beta E_l/2)$), all in terms of the u's and v's. Note also that the BCS ground state solution is a coherent state of Cooper pairs and does not have a fixed number of particles. Hence N is the only free parameter in this problem! *Hint: you can look all of these things up from the lecture summaries posted on the class website.*

b) Starting with Eq. (1) of the Lecture 9 Summary, do a variational calculation to find the BCS-like ground state (i.e. the u's and v's) for this problem, making the BCS approximation of a single value $-V$ for the coupling matrix element between electrons in all M states. (*Hint: In doing this, it is better to start from scratch using BCS ideas, and not to try to take the limiting forms of the BCS results. Note that the level degeneracy implies that all states play an equal role, and there is no need to make anything k-dependent.*) The answer is $u = v = 1/\sqrt{2}$.

c) Now derive a more general result for the u's and v's. Combine the average number of particles (N) constraint given above, and the normalization condition ($|u|^2 + |v|^2 = 1$), to find that $u = \sqrt{1 - N/2M}$ and $v = \sqrt{N/2M}$. Recall that the number of Cooper pairs in the BCS ground state wavefunction (a coherent state of Cooper pairs) is not fixed. Hence N is the only variable in this problem. What value of N will make the results of parts (b) and (c) equal?

d) Starting from the definition of the gap function Δ_k given above, and the u's and v's from part (c), find the gap parameter $\Delta(0)$ at $T = 0$ in terms of M, N and V. For what values of N/M is it a maximum? What is that maximum? Sketch $\Delta(0)$ vs. N/M for values between

0 and 2. This plot has some resemblance to the T_c vs. doping (p) curve in the high-T_c cuprates, which looks like an inverted parabola:



e) Specializing to the case of a half-filled band, $N = M$, find the expectation value of the Landau potential (from Lecture 9) at $T = 0$, in terms of M and V.

f) For the case $N = M$, use the finite-temperature BCS gap equation given above to set up a simple transcendental equation determining $\Delta(T)$, and analytically determine T_c, the temperature at which $\Delta(T)$ goes to zero. Note that T_c is independent of $\hbar\omega_c$ in this model, so that there is no isotope effect. Find the ratio $\Delta(0)/k_BT_c$, and compare it with the BCS value of 1.76.

g) Using results from the parts above, show that the self-consistent gap equation can be written as $\frac{\Delta(T)}{\Delta(0)} = \tanh\left(\frac{\Delta(T)}{\Delta(0)}\frac{T_c}{T}\right)$. By numerical means, plot $\Delta(t)/\Delta(0)$ as a function of $t = T/T_c$. (This problem is mathematically equivalent to the determination of the molecular field solution for an $S = 1/2$ ferromagnet.) By suitable expansions of the transcendental equation, find analytic limiting forms for $\Delta(t)/\Delta(0)$ valid near $t = 0$ and $t = 1$. Compare the high-temperature limit with the BCS result that $[\Delta(t)/\Delta(0)]^2 = 3.016\,(1 - t)$.

2. **BCS Ground State Energy**

In the calculation of the energy difference between the superconducting state and the normal state at zero temperature in lecture 10, $\langle E\rangle_{SC} - \langle E\rangle_N$, we come to this expression:

$$\langle E\rangle_{SC} - \langle E\rangle_N = 2\sum_{k>k_F}\left(\xi_k - \frac{\xi_k^2}{E_k}\right) - \frac{\Delta^2}{V}$$

By converting the sum on k to an integral on energy ξ and carrying out the integral, show that the final result is:

$$\langle E\rangle_{SC} - \langle E\rangle_N \cong \left(\frac{\Delta^2}{V} - \frac{D(0)\Delta^2}{2}\right) - \frac{\Delta^2}{V}$$

where we have assumed that the density of states $D(\xi)$ is roughly constant over the range of integration, and that $\frac{\Delta}{\hbar\omega_c} \ll 1$. *Hints: i) Note from the Lecture 9 Summary that* $\Delta_k = 0$ *for* $\xi_k > \hbar\omega_c$*, which has consequences for the integrand, ii) When you evaluate the energy* *integral, make sure to use the* $\sinh^{-1}(\cdot)$ *form for the answer, and iii) judiciously utilize the fact that* $\frac{\Delta}{\hbar\omega_c} = \frac{1}{\sinh(1/D(0)V)}$.